Low-Rank Block Compression in Sparse-Grid Methods

Context

The so-called *curse of dimensionality* [1] refers to the rapid increase in computational cost as the dimension of the problem grows. Traditional numerical methods, such as full tensor-product (full-grid) discretizations, quickly become infeasible in high-dimensional settings because their computational complexity grows exponentially with the dimension. Sparse-grid methods [2] offer an efficient way to mitigate this issue. They are based on a truncation of tensor products in multiresolution spaces, which allows a substantial reduction in computational cost while preserving good approximation accuracy. Under suitable assumptions, problems that would normally require a cost proportional to $\mathcal{O}(\varepsilon^{-d})$ to achieve an accuracy of order $\mathcal{O}(\varepsilon^{p})$ can instead be solved with a complexity of only $\mathcal{O}(\varepsilon^{-1}|\log\varepsilon|^{d-1})$ using sparse grids, thus avoiding —actually up to a logarithmic factor— the exponential dependence on the dimension d, see Fig. 1 for an illustration in two dimensions.

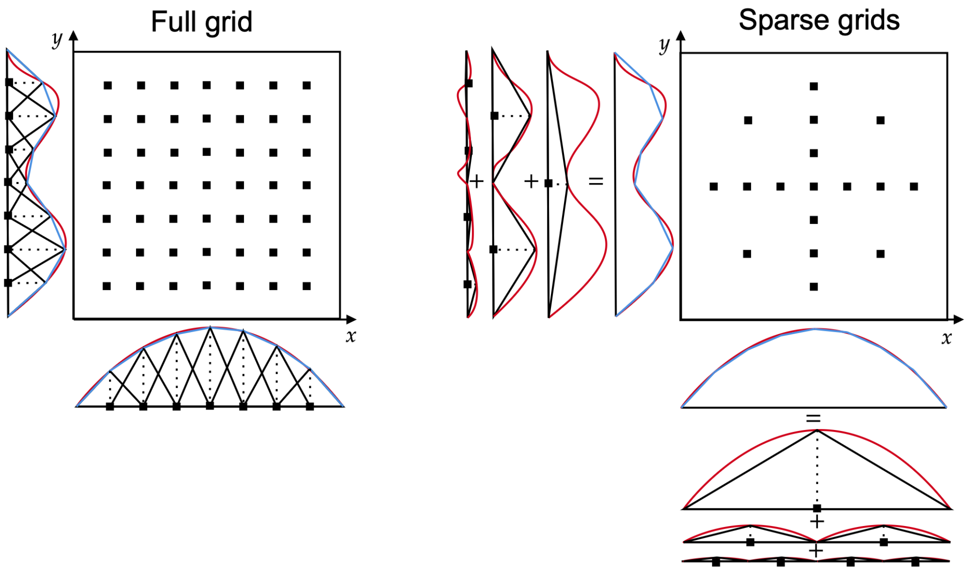

Figure 1: Illustration of full-grid and sparse-grid approximations for a two-dimensional function $f(x, y)$.

Objective

The goal of this internship is to study the combination of sparse-grid methods with compression techniques for approximating the solution u of a partial differential equation (PDE). As a first example, we will consider the Poisson problem with homogeneous Dirichlet boundary conditions:

$$\Delta u = f \quad \text{in } \Omega = [0,1]^2, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{2}$$

Using a Galerkin method and a discretization based on piecewise polynomial functions and sparse-grid tensor products, this PDE leads to a linear system $\mathbf{Ax} = \mathbf{b}$. Because of the hierarchical multiresolution structure inherent to sparse grids, the matrix $\mathbf{A}$ exhibits a specific block structure composed of both sparse and dense parts. Some of these dense sub-blocks can be represented efficiently using *low-rank approximations*. The main objective of the internship is to investigate how low-rank compression techniques can be combined with sparse-grid discretizations to further reduce computational cost and memory requirements.

In addition, the hierarchical structure of the matrix $\mathbf{A}$ provides insight into the relative importance of the contributions in the approximation. This information can be further exploited to reduce computational cost. Another objective of the internship is therefore to investigate the use of *mixed-precision arithmetic*, assigning lower numerical precision to matrix blocks that contribute less significantly to the overall accuracy of the approximation.

The internship will involve the following main steps: understanding the structure of sparse-grid discretizations and assembling the corresponding linear system for the Poisson problem Eq. (1)-Eq. (2); identifying different classes of blocks in the resulting matrix $\mathbf{A}$, depending on whether low-rank compression, mixed-precision arithmetic, both, or neither should be applied; investigating various low-rank approximation and compression techniques; and implementing and testing the selected methods on representative examples, such as the model problem Eq. (1)-Eq. (2) and other applications.

Practical Information

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References

[1] R. E. Bellman. *Adaptive Control Processes: A Guided Tour.* Princeton University Press, Princeton, NJ, 1961.

[2] B. Hans-Joachim and G. Michael. Sparse grids. *Acta Numerica*, 13:147–269, May 2004.

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